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## COMPARISONS OF METHODS OF COMPUTING BENDING MOMENTS IN HELICOPTER ROTOR BLADES IN THE PLANE OF FLAPPING

By John E. Duberg and Arthur R. Luecker

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

NACA

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ADVANCE RESTRICTED REPORT

# COMPARISONS OF METHODS OF COMPUTING BENDING MOMENTS IN HELICOPTER ROTOR BLADES IN THE PLANE OF FLAPPING

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## SUMMARY

Several methods of computing bending moments in helicopter rotor blades in the plane of flapping are reviewed, and the results of a numerical example analyzed by four different methods are compared. The effect on computed bending moments of the tip-loss correction introduced by Wheatley is also considered.

The comparison indicates that, from the standpoint of accuracy of results and ease of application, the method proposed by Cierva is the most suitable for routine analyses. The tip-loss correction is shown to have a substantial effect on computed bending moments.

## INTRODUCTION

The determination of the bending stresses in rotor blades during flight is one of the important problems in the stress analysis of the structure of the helicopter. The problem is complicated by the fact that the air loads and the inertia loads are continuously changing as the blades rotate and that the bending deflection of the blades has an important effect on the moments developed therein. The analysis can be simplified if the blades are assumed rigid and therefore unable to bend, but under this assumption the computed bending stresses superimposed on the centrifugal-tension stresses are relatively high. A more exact analysis, in which the relieving effects of centrifugal tension on blade bending are included, gives much lower values for these calculated bending stresses.

In the present paper comparisons are made between the various methods that have been proposed for the calculation of the bending stresses in the plane of flapping. The effect of introducing a correction for tip loss, which has heretofore been ignored in stress analyses, is also considered. A numerical example is analyzed by each of the several methods to provide a concrete basis for the comparisons.

## THEORY AND BASIC ASSUMPTIONS OF METHODS OF ANALYSIS

The basic rotor theory on which stress analyses of rotor blades are based is that of Glauert, Lock, and Wheatley. (See references 1, 2, and 3, respectively.) A consideration of the forces acting on the rotor blade during steady forward flight, as given by this theory, indicates that the total bending moment at any radial station may be resolved into the following three components:

(1) A component that is identical with the bending moment in a rigid blade and is therefore independent of blade bending

(2) A component, due to axial tension, that depends on the blade deflection and therefore varies as the blade bends

(3) A component due to the inertia forces associated with the variation of the deflection with time

In the simplified theories, in which the blade is assumed static and rigid, the second and third components do not occur. When flexibility is taken into account so that the second component is included, the problem is complicated because of the interaction between the moments and the blade deflections. If the third component is included, the problem is further complicated because the elastic curve must satisfy dynamic as well as static conditions.

## METHODS OF ANALYSIS

The methods of analysis for rotor-blade bending moments in the plane of flapping that are available in the literature fall into three categories as follows:

(1) "Exact" analyses in which inertia forces due to rate of change of blade deflections, as well

as the interaction between blade deflections and moments, are considered

(2) Analyses, based on static loadings, that include interaction between blade bending and moments but neglect inertia forces due to blade-bending deflections

(3) Simplified analyses in which the bending moments for the rigid blade are computed and approximate corrections to account for the reduction in the bending moment due to the axial load are applied; the reduction due to the axial load has sometimes been called centrifugal relief

The analysis given in reference 4 is believed to be the only published exact analysis and is based on the rotor theory of references 1, 2, and 5. This theory has been refined and extended in reference 3. In appendix A of the present paper, one of these refinements - allowance for the reduction in lift near the blade tip - is introduced into the analysis of blade bending.

Most of the solutions that have been presented in the literature fall into the second category, in which the problem treated is purely a static one. This approach reduces the problem to that of a beam under combined bending and tension and, therefore, the methods developed for such problems can be applied to the analysis of rotor-blade moments. In reference 6, a solution of this class, which makes use of "type" solutions to facilitate the numerical calculations, is given as a simplification of the analysis of reference 4. Two other methods of solving the static problem are given in references 7 and 8. In reference 7 the solution is effected by means of the theorem of three moments generalized to include the effect of centrifugal tension. In reference 8 the solution is obtained by finding a deflected shape for the blade that is consistent with a minimum of potential energy for the system.

The methods of the third category, which should have more appeal for use in practical applications, provide simplified formulas for obtaining the bending moments in the blade by correcting the rigid-blade bending moments for the relief caused by centrifugal tension. Cierva has proposed a formula, given in reference 6, which states that the moment developed in the actual rotor blade is equal to the product of the rigid-blade bending moment and the perfectly flexible blade bending moment divided by the sum of these two moments. The perfectly flexible blade bending moment is a fictitious bending moment obtained by multiplying the stiffness of the actual blade by the curvature of the perfectly flexible blade. The solutions for the rigid-blade bending moment and the flexible-blade bending moment are given in appendix B. Hohenemser in reference 9 has developed a method of computing the blade bending moment by multiplying the rigid-blade moment by a correction factor, given herein in appendix B, that depends on the blade radius, blade stiffness, and centrifugal force at the root. The formula is developed on the basis of uniform distribution along the length of the blade of both the blade mass and blade bending stiffness. Reference 10 contains a brief discussion of Hohenemser's formula as well as the development of formulas for the calculation of the rigid-blade bending moments.

Although wind-tunnel and flight tests have proved the basic rotor theory reliable when applied to the rotor as a whole, the theory contains approximations that raise some doubts as to the order of accuracy in computing air forces at particular points in the rotor disk and, hence, doubts as to the accuracy of bending moments computed by any of the methods mentioned herein. These doubts can be removed only by further tests in which blade bending is measured under operating conditions.

## COMPARISON OF BENDING MOMENTS COMPUTED BY DIFFERENT METHODS

In order to compare the bending moments obtained by the several methods of analysis, calculations were made for a rotor with three blades, each having the following physical properties:

| | |
|---|---|
| Radius, feet . . . . . . . . . . . . . . . . . . . . | 12.5 |
| Mass, slug per foot . . . . . . . . . . . . . . . . | 0.0519 |
| Blade chord, inches . . . . . . . . . . . . . . . . | $9\frac{1}{2}$ |
| Bending stiffness, EI, pound-feet$^2$ . . . . . . . . | 7640 |
| Pitch setting (untwisted), degrees . . . . . . . . . | 10 |

The rotor was assumed to be rotating at 370 rpm and to have a forward velocity of 100 miles per hour, which corresponds to a tip-speed ratio $\mu$ of 0.30. The ratio $\lambda$ of axial velocity to tip speed was assumed to be -0.079; the negative sign indicates flow downward through the rotor disk.

The bending moments in the plane of flapping have been computed for the rotor blade described in the preceding paragraph by methods in each of the three catagories of analysis mentioned in the preceding section. The moments computed for four azimuth angles ($\psi$) are presented in figure 1. Inspection of these moment diagrams reveals that only a very small difference in computed bending moment at any station exists between the values given by the exact method and this exact method modified by neglecting the inertia forces due to blade bending. The small magnitude of this difference substantiates the assumption, common to all approximate methods of analysis, that these inertia forces are negligible.

The method of analysis proposed by Cierva, which is given in detail in appendix B, gives a maximum bending moment for any azimuth angle that differs only slightly from the maximum bending moment given by the exact method for the same azimuth angle. The moment diagrams given by the Cierva method are in close agreement with the moment diagrams for the exact method except for the outer third of the blade. In the outer third of the blade the bending moments predicted by the Cierva formula decrease to zero less rapidly than those predicted by the exact method, but this fact is of little consequence because in this part of the blade the moments are less than the maximum moment.

The method of analysis proposed by Hohenemser gives, as shown in figure 1, a bending moment diagram at each azimuth angle that differs appreciably from those given by the other methods. The computed moment is higher in the inner part of the blade and lower in the outer part as compared with the moment computed by the exact method. At an azimuth angle $\psi$ of 90°, where the maximum bending moment approaches its smallest value, the Hohenemser method gives a maximum bending moment appreciably higher than that given by the exact method. At $\psi = 270°$, where the maximum bending moment approaches its largest value, the Hohenemser method gives a maximum bending moment appreciably smaller than that given by the exact method.

The labor required to evaluate the bending stress by the approximate methods of Cierva and Hohenemser is only a small fraction of the labor required by the more exact methods. Of these two approximate methods, that of Cierva is recommended for practical stress analysis of helicopter blades because the bending stresses obtained thereby are in good agreement with those given by the exact method.

In figure 2 are shown the results obtained by the exact analysis extended to include the effect of tip loss by the method introduced by Wheatley (reference 3). The tip-loss factor was assumed to be 0.97, which means that the outer 3 percent of the blade is assumed to produce no lift. The method of analysis used assumes zero moment at the 0.97 point and does not consider bending between this point and the tip. At all azimuth angles, the consideration of tip loss reduces the maximum bending moment considerably and causes a reversal of bending near the tip. Because the assumption made concerning tip effect is only a crude approximation of the actual lift distribution in this region, the calculated moments near the tip may be very much in error. The large reduction in the bending moment near the center, however, is significant.

The bending stresses at each station in the rotor blades of a helicopter vary between a maximum and a minimum with each revolution of the rotor. For the particular blade analyzed, which was assumed to be of all-metal aluminum-alloy construction with a section modulus of 0.167 inch cubed and an effective cross-sectional area of 0.56 square inch, the total stress, which includes the centrifugal tension and the superposed bending stress, varied between the limits shown in figure 3 for the upper and lower fibers at the different stations along the blade. Figure 3 is based on the results of the exact bending-moment analysis, neglecting tip loss.

## CONCLUSIONS

A comparison was made of rotor blade bending-moments obtained by several methods of analysis. The results for the single numerical example studied indicate that the additional accuracy obtainable by the "exact" analysis, when compared with the best of the approximate methods now in use, does not justify the extra labor involved in applying the exact method.

Of the two approximate methods studied, that of Cierva gives the best over-all agreement with the more exact analysis and is well suited for the determination of stresses in blades. The method proposed by Hohenemser, although easier to apply than the Cierva method, is less desirable because it gives maximum bending moments that may differ by as much as one-quarter from those computed by the exact method.

Consideration of tip loss results in a substantial reduction in maximum bending moments and should receive further attention in future studies of methods of blade stress analysis.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., Ju

# APPENDIX A

## EXACT METHOD OF CALCULATING ROTOR-BLADE BENDING MOMENTS

### Differential Equation of Blade Bending

The basic assumptions of the blade-bending analysis referred to herein as the exact method are identical with those used in reference 4 for the development of the differential equation except that a correction is added to include the effect of tip loss. The correction for tip loss follows the method developed by Wheatley in reference 3 in which the air load on the outer few percent of the blade is neglected.

Figure 4 shows the coordinate system used and the nomenclature involved in defining the deviation of the flexible blade from the rigid blade. The symbols used in the analysis are defined in appendix C. In figure 5 are shown the forces and moments acting on each blade element. A consideration of the equilibrium of the element in the direction parallel to the longitudinal axis of the blade and in the direction normal to the blade yields the following differential equation for the blade bending deflections:

$$\frac{d^4y}{dx^4} - K\left(1 + \frac{2QR_Q}{mB^2R^2} - x^2\right)\frac{d^2y}{dx^2} + 2Kx\frac{dy}{dx} + \frac{2K}{\Omega^2}\frac{d^2y}{dt^2} = \frac{B^4R^4}{EI}\frac{dS}{dr} \quad \text{(A1)}$$

If the blade mass is uniformly distributed from hinge to tip, the mass of the tip section, which is assumed to be without air load, is

$$Q = mR(1 - B)$$

and the distance from the tip mass to the hinge is

$$R_Q = \frac{R}{2}(1 + B)$$

If these relationships are substituted in the second term of the differential equation some simplification results and the differential equation becomes

$$\frac{d^4y}{dx^4} - K\left(\frac{1}{B^2} - x^2\right)\frac{d^2y}{dx^2} + 2Kx\frac{dy}{dx} + \frac{2K}{\Omega^2}\frac{d^2y}{dt^2} = \frac{B^4R^4}{EI}\frac{dS}{dr}$$

If the tip loss is neglected, B reduces to unity and the differential equation reduces to that given in reference 4.

## Blade Loading

The term on the right-hand side of equation (A1) represents the blade load gradient, which is assumed to be the same as that on a rigid blade. The blade loading can be evaluated by means of the rotor theory given in references 1 and 3. In the present analysis the blades are assumed to be built with constant chord and without twist and the effect of periodic twist of the blades is neglected. The effect of both types of twist can be considered by the methods of references 2 and 3.

If the instantaneous flapping angle of the blades is represented by the equation

$$\beta = a_o - a_1 \cos \psi - b_1 \sin \psi \tag{A2}$$

the velocity components causing aerodynamic forces on the blades are

$$\left.\begin{aligned} U_T &= xBR\Omega + \mu R\Omega \sin \psi \\ U_P &= \lambda R\Omega - xBR\Omega\left(a_1 \sin \psi - b_1 \cos \psi\right) \\ &\quad -\mu R\Omega\left(a_o - a_1 \cos \psi - b_1 \sin \psi\right)\cos \psi \end{aligned}\right\} \tag{A3}$$

The load gradient is then given by

$$\frac{dS}{dr} = \frac{1}{2}c\rho a\left(U_T U_P + \theta U_T^2\right) - mBR\Omega^2 a_o x - mg \tag{A4}$$

Introduction of the relationships given in equations (A3) into equation (A4), expanding, and neglecting higher harmonics gives for the load gradient

$$\frac{dS}{dr} = \frac{1}{2}c\rho aR^2\Omega^2 \left\{\left[\theta B^2x^2 + B\left(\lambda - \frac{2a_0m}{c\rho aR}\right)x + \frac{1}{2}\theta\mu^2 - \frac{2mg}{c\rho aR^2\Omega^2}\right] + \left[-a_1B^2x^2 + 2\theta\mu Bx + \mu\lambda + \frac{1}{4}\mu^2a_1\right]\sin\psi + \left[b_1B^2x^2 - a_0\mu Bx + \frac{1}{4}\mu^2b_1\right]\cos\psi\right\} \tag{A5}$$

Flapping Coefficients

The flapping coefficients $a_0$, $a_1$, and $b_1$, which define the blade motion given in equation (A2), are determined from the condition that the moment at the flapping hinge is zero for all azimuth angles. The coefficients are:

$$\left.\begin{aligned} a_0 &= \gamma\left[\frac{1}{8}\theta B^2\left(B^2 + \mu^2\right) + \frac{1}{6}B^3\lambda\right] - \frac{M_W}{\Omega^2 I_1} \\ a_1 &= \frac{4\mu(4B\theta + 3\lambda)}{3\left(2B^2 - \mu^2\right)} \\ b_1 &= \frac{8\mu Ba_0}{3\left(2B^2 + \mu^2\right)} \end{aligned}\right\} \tag{A6}$$

where

$$M_W = g\left(QBR + \frac{1}{2}mB^2R^2\right)$$

$$I_1 = QBRR_Q + \frac{1}{3}mB^3R^3$$

$$\gamma = \frac{c\rho aR^4}{I_1}$$

## Solution of the Differential Equation of Blade Bending

The method of solution of the differential equation follows that given in reference 4. The blade deflections are assumed to be expressible in the form

$$y = y_1 + y_2 \sin \psi + y_3 \cos \psi \qquad \text{(A7)}$$

in which $y_1$, $y_2$, and $y_3$ are functions of the distance along the blade span. Substitution of equation (A7) and (A5) in the differential equation (A1) yields the following differential equations for the functions $y_1$, $y_2$, and $y_3$.

$$\frac{d^4y_1}{dx^4} - K\left[\left(1 + \frac{2QR_Q}{mB^2R^2}\right) - x^2\right]\frac{d^2y_1}{dx^2} + 2Kx\frac{dy_1}{dx} = \frac{c\rho aB^4R^6\Omega^2}{2EI}\left[\theta B^2x^2 + \left(\lambda - \frac{2a_0m}{c\rho aR}\right)Bx + \frac{1}{2}\left(\theta\mu^2 - \frac{4mg}{c\rho aR^2\Omega^2}\right)\right] \qquad \text{(A8)}$$

$$\frac{d^4y_2}{dx^4} - K\left[\left(1 + \frac{2QR_Q}{mB^2R^2}\right) - x^2\right]\frac{d^2y_2}{dx^2} + 2Kx\frac{dy_2}{dx} - 2Ky_2 = \frac{c\rho aB^4R^6\Omega^2}{2EI}\left[-a_1B^2x^2 + 2\theta\mu Bx + \left(\mu\lambda + \frac{1}{4}\mu^2a_1\right)\right] \qquad \text{(A9)}$$

$$\frac{d^4y_3}{dx^4} - K\left[\left(1 + \frac{2QR_Q}{mB^2R^2}\right) - x^2\right]\frac{d^2y_3}{dx^2} + 2Kx\frac{dy_3}{dx} - 2Ky_3 = \frac{c\rho aB^4R^6\Omega^2}{2EI}\left[b_1B^2x^2 - a_0\mu Bx + \frac{1}{4}\mu^2b_1\right] \qquad \text{(A10)}$$

The four boundary conditions that the functions $y_1$, $y_2$, and $y_3$ must satisfy are : At the hinge the deflection and moment are zero, and at radius BR the moment is zero and the shear is that required to maintain equilibrium of the tip mass Q. The last-mentioned boundary condition, at $x = 1.0$, can be expressed in the following form:

$$A\frac{d^3y_1}{dx^3} = \frac{dy_1}{dx} + BRa_o + \frac{BRg}{R_Q\Omega^2}$$

$$A\frac{d^3y_2}{dx^3} = \frac{dy_2}{dx} - y_2$$

$$A\frac{d^3y_3}{dx^3} = \frac{dy_3}{dx} - y_3$$

where

$$A = \frac{EI}{Q\Omega^2 B^2R^2R_Q}$$

The differential equations for $y_1$, $y_2$, and $y_3$ can be solved approximately by the method of "collocation". (See reference 4.) The method of collocation consists essentially of expressing the solution as a linear combination of functions that satisfies the boundary conditions independently of the value of the coefficient associated with each function. The combination of functions is substituted into the differential equation, and the coefficients are so adjusted that the resultant expression satisfies the differential equation at as many points as there are functions.

When the tip effect is considered the following set of functions can be used:

$$y_1 = -\left(BRa_o + \frac{BRg}{R_Q\Omega^2}\right) x + \sum_{p=1}^{n} C_{1p}\left\{x^{p+2} - \frac{p+1}{p+3}x^{p+3} - \left[(p+1)(p+2)A + 1\right]x\right\}$$

$$y_2 = \sum_{p=1}^{n} C_{2p}\left(x^{p+2} - \frac{p + 1}{p + 3}x^{p+3}\right)$$

$$y_3 = \sum_{p=1}^{n} C_{3p}\left(x^{p+2} - \frac{p + 1}{p + 3}x^{p+3}\right)$$

If the tip effect is neglected, the form of the solution for $y_1$ is

$$y_1 = C_{11}x + \sum_{p=2}^{n} C_{1p}\left(x^{p+1} - \frac{2p}{p + 2}x^{p+2} + \frac{p(p + 1)}{(p + 2)(p + 3)}x^{p+3}\right)$$

The form of solution for $y_2$ and $y_3$ remains as before.

## Numerical Example

A numerical example is presented with tip loss neglected, for a blade with the following properties and operating conditions:

$R = 12.5$ feet

$c = 9\frac{1}{2}$ inches

$m = 0.0519$ slug per foot

$a = 5.73$

$EI = 7640$ pound-feet$^2$

$\Omega = 38.8$ radians per second

$\mu = 0.300$

$\lambda = -0.079$

$\theta = 10^\circ = 0.175$ radian

$\rho = 0.00230$ slug per cubic foot

If the tip loss is neglected,

$$B = 1.0 \text{ and } Q = 0$$

Equations (A6) give the following values for the flapping coefficients:

$$a_o = 0.077922$$

$$a_1 = 0.096963$$

$$b_1 = 0.029827$$

The factors $\gamma$ and $K$ have the following values:

$$\gamma = 7.5385$$

$$K = 124.83$$

If six functions are chosen for the solution to the differential equation for $y_1$, the form of the equation is

$$y_1 = C_{11}x + C_{12}\left(x^3 - x^4 + \frac{3}{10}x^5\right) + C_{13}\left(x^4 - \frac{6}{5}x^5 + \frac{2}{5}x^6\right)$$
$$+ C_{14}\left(x^5 - \frac{4}{3}x^6 + \frac{10}{21}x^7\right) + C_{15}\left(x^6 - \frac{10}{7}x^7 + \frac{15}{28}x^8\right)$$
$$+ C_{16}\left(x^7 - \frac{3}{2}x^8 + \frac{7}{12}x^9\right) \tag{A11}$$

When this function is substituted in the differential equation (A8), the equation is satisfied at the points $x = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0 if

$C_{11} = -0.171$ foot $\quad$ $C_{14} = -0.919$ foot

$C_{12} = 0.133$ foot $\quad$ $C_{15} = 0.569$ foot

$C_{13} = 1.197$ feet $\quad$ $C_{16} = 0.503$ foot

Substitution of the coefficients in equation (A11) yields

$$y_1 = -0.171x + 0.133x^3 + 1.064x^4 - 2.315x^5 + 2.273x^6$$

$$-0.748x^7 - 0.449x^8 + 0.293x^9$$

The corresponding bending moments are obtained by differentiating twice and multiplying by $\frac{EI}{R^2}$, which gives

$$M_1 = 48.9(0.795x + 12.77x^2 - 46.30x^3 + 68.18x^4$$

$$-31.40x^5 - 25.17x^6 + 21.12x^7)$$

By a similar process the results for $M_2$ and $M_3$ corresponding to the deflections $y_2$ and $y_3$ are

$$M_2 = 48.9(5.16x - 42.19x^2 + 136.1x^3 - 239.9x^4$$

$$+ 232.3x^5 - 109.2x^6 + 17.8x^7)$$

$$M_3 = 48.9(0.333x + 1.32x^2 - 5.08x^3 + 5.93x^4$$

$$+ 1.59x^5 - 9.18x^6 + 5.09x^7)$$

The general expression for the bending moments is

$$M = M_1 + M_2 \sin\psi + M_3 \cos\psi$$

At a distance from the flapping hinge of $x = 0.6$ the bending moment is

$$M = 43.2 - 15.9 \sin\psi + 9.0 \cos\psi$$

The minimum bending moment at the station $x = 0.6$ occurs when $\psi = 120^\circ$ and is

$$M = 43.2 - 13.8 - 4.5 = 24.9 \text{ pound-feet}$$

The maximum bending moment at the station $x = 0.6$ occurs when $\psi = 300^\circ$ and is

$$M = 43.2 + 13.8 + 4.5 = 61.5 \text{ pound-feet}$$

# APPENDIX B

## APPROXIMATE METHODS OF CALCULATING ROTOR-BLADE BENDING MOMENTS

### Cierva Method

The Cierva formula for the bending moment in a rotor blade, given in reference 6, is

$$M = \frac{M_r M_f}{M_r + M_f} \tag{B1}$$

where $M_r$ is the moment in a rigid blade and $M_f$ is a fictitious moment obtained by multiplying the curvature of a perfectly flexible blade by the actual blade bending stiffness EI.

If the blade is assumed rigid and tip loss is neglected, the rigid-blade bending moment obtained by integrating the moment of the load gradient given in equation (A5) is

$$\begin{aligned} M_r = \frac{1}{2}c\rho aR^4\Omega^2 \Bigg\{ & \left[\theta C_x + \left(\lambda - \frac{6a_o}{\gamma}\right)B_x + \left(\frac{1}{2}\theta\mu^2 - \frac{6g}{\gamma R\Omega^2}\right)A_x\right] \\ & + \left[-a_1C_x + 2\theta\mu B_x + \left(\mu\lambda + \frac{\mu^2 a_1}{4}\right)A_x\right]\sin\psi \\ & + \left[b_1C_x - \mu a_o B_x + \frac{\mu^2 b_1}{4}A_x\right]\cos\psi \Bigg\} \end{aligned} \tag{B2}$$

in which

$$\left.\begin{aligned} A_x &= \int_x^1 \int_x^1 dxdx = \frac{1}{2}\left(1 - 2x + x^2\right) \\ B_x &= \int_x^1 \int_x^1 xdxdx = \frac{1}{6}\left(2 - 3x + x^3\right) \\ C_x &= \int_x^1 \int_x^1 x^2dxdx = \frac{1}{12}\left(3 - 4x + x^4\right) \end{aligned}\right\} \tag{B3}$$

If the blade loading for the perfectly flexible blade is assumed equal to that of the rigid blade the fictitious moment existing in the blade is

$$M_f = \frac{EI\gamma}{9R}\left\{\left[\theta - \frac{\theta + \frac{3}{2}\theta\mu^2 - \frac{18g}{\gamma R\Omega^2}}{(1 + x)^2}\right] + \left[-a_1 - \frac{-a_1 + \frac{3}{4}a_1\mu^2 + 3\mu\lambda}{(1 + x)^2}\right]\sin\psi + \left[b_1 - \frac{b_1 + \frac{3}{4}b_1\mu^2}{(1 + x)^2}\right]\cos\psi\right\} \tag{B4}$$

The numerical values of $M_f$ and $M_r$ can then be combined to give the actual blade bending moment according to equation (B1).

In reference 9 Hohenemser developed the following formula for the moment in a blade that has a uniform distribution of both blade mass and blade bending stiffness:

$$M = \frac{M_r}{1 + 0.052\frac{R^2P_o}{EI}} \tag{B5}$$

in which $M_r$ is the rigid-blade bending moment and $P_o$ is the centrifugal tension at the flapping hinge.

For a blade of uniform mass distribution,

$$P_o = \frac{1}{2}mR^2\Omega^2$$

Substitution of the value for $P_o$ in equation (B5) reduces this equation to

$$M = \frac{M_r}{1 + 0.052K}$$

in which

$$K = \frac{mR^4\Omega^2}{2EI}$$

Numerical Examples

The bending moments in the same blade and for the same operating conditions as were considered in appendix A are calculated by both Cierva's and Hohenemser's method for $x = 0.6$ and $\psi = 120^\circ$ and $300^\circ$.

When the basic data from appendix A are substituted in equation (B2) the following result is obtained:

$$\begin{aligned} M_r = 192{,}000\Big[&(0.1750C_x - 0.1410B_x + 0.00651A_x) \\ &- (0.0970C_x - 0.1050B_x + 0.0215A_x)\sin\psi \\ &+ (0.0298C_x - 0.0234B_x + 0.00067A_x)\cos\psi\Big] \end{aligned} \qquad \text{(B6)}$$

For $x = 0.6$, equations (B3) give

$$A_x = 0.0800$$

$$B_x = 0.0693$$

$$C_x = 0.0608$$

Substitution of these values of $A_x$, $B_x$, and $C_x$ in equation (B6) gives the following expression for the rigid-blade moment at $x = 0.6$:

$$M_r = 265 - 65 \sin\psi + 48 \cos\psi$$

The perfectly flexible blade bending moment is obtained from equation (B4) and is

$$\begin{aligned} M_f = 512\Bigg\{&\left[0.1750 - \frac{0.1945}{(1 + x)^2}\right] + \left[-0.0970 + \frac{0.1615}{(1 + x)^2}\right]\sin\psi \\ &+ \left[0.0298 - \frac{0.0318}{(1 + x)^2}\right]\cos\psi\Bigg\} \end{aligned}$$

At $x = 0.6$,

$$M_f = 50.7 - 17.4 \sin \psi + 8.9 \cos \psi$$

At $\psi = 120^\circ$ (the approximate position for minimum bending moment),

$$M_r = 265 - 56 - 24 = 185 \text{ pound-feet}$$

$$M_f = 50.7 - 15.1 - 4.4 = 31.2 \text{ pound-feet}$$

Therefore, the Cierva method (see equation (B1)) gives for the bending moment $M$ at the azimuth position $\psi = 120^\circ$,

$$M = \frac{185 \times 31.2}{185 + 31} = 26.7 \text{ pound-feet}$$

Similarly, the Cierva method gives for $\psi = 300^\circ$

$$M = 58.4 \text{ pound-feet}$$

When Hohenemser's method is used, at $\psi = 120^\circ$,

$$M = 185 \times \frac{1}{1 + (0.052 \times 124.8)} = 24.7 \text{ pound-feet}$$

and similarly, at $\psi = 300^\circ$,

$$M = 345 \times \frac{1}{1 + (0.052 \times 124.8)} = 46.0 \text{ pound-feet}$$

## APPENDIX C

## SYMBOLS

| | |
|---|---|
| $A$ | dimensionless coefficient $\left(EI/QB^2R^2R_Q\Omega^2\right)$ |
| $a$ | slope of lift curve |
| $a_o$ | constant term in Fourier series that expresses $\beta$ |
| $a_1$ | coefficient of $\cos\psi$ in expression for $\beta$ |
| $b_1$ | coefficient of $\sin\psi$ in expression for $\beta$ |
| $B$ | tip-loss factor (blade elements outboard of radius $BR$ are assumed to have no lift) |
| $C_{1p}, C_{2p}, C_{3p}$ | coefficients in equations for $y_1$, $y_2$, and $y_3$ |
| $c$ | blade chord (constant) |
| $EI$ | flexural stiffness of blade |
| $g$ | acceleration due to gravity |
| $I_1$ | mass moment of inertia of one rotor blade about horizontal hinge |
| $K$ | dimensionless coefficient $\left(mB^4R^4\Omega^2/2EI\right)$ |
| $dL$ | aerodynamic lift on blade element at radius $r$ |
| $M$ | bending moment in blade at radius $r$ |
| $M_1$, $M_2$, $M_3$ | moments corresponding to the deflection functions $y_1$, $y_2$, and $y_3$ |
| $M_f$ | flexible-blade bending moment as defined in appendix B |
| $M_r$ | bending moment in blade at radius $r$ (blade assumed to be a rigid body) |
| $M_W$ | weight moment of blade about horizontal hinge |

| | |
|---|---|
| $m$ | mass of blade per unit length between hinge and radius $BR$ |
| $n$ | an arbitrary integer |
| $p$ | any integer greater than zero and less than or equal to $n$ |
| $P$ | tension in blade at radius $r$ |
| $P_0$ | tension in blade at horizontal hinge |
| $Q$ | mass of blade tip between radius $BR$ and radius $R$ |
| $R$ | blade radius |
| $R_Q$ | distance from center of rotation to center of gravity of mass $Q$ |
| $r$ | radius of blade element |
| $S$ | shear in the blade at radius $r$ |
| $t$ | time |
| $U_T$ | velocity component at blade element perpendicular to blade span axis and parallel to rotor disk |
| $U_P$ | velocity component at blade element perpendicular both to blade span and to $U_T$ |
| $x$ | ratio of blade-element radius to $BR$ |
| $y$ | deflection of blade element at radius $r$, referred to rigid-blade position |
| $y_1$, $y_2$, $y_3$ | deflection functions entering into the general equation for $y$ |
| $\beta$ | blade flapping angle |
| $\beta'$ | angle between plane perpendicular to axis of rotation and line connecting horizontal hinge with blade element at radius $r$ |

| | |
|---|---|
| $\phi$ | slope of tangent to blade at radius $r$, referred to plane perpendicular to axis of rotation |
| $\psi$ | blade azimuth angle, measured in direction of rotation from down-wind position |
| $\theta$ | blade pitch angle |
| $\Omega$ | angular velocity of rotor |
| $\mu$ | ratio of component of forward speed in plane perpendicular to axis of rotation to $\Omega R$ |
| $\lambda$ | ratio of axial inflow velocity through rotor to $\Omega R$ |
| $\gamma$ | mass constant of rotor blade $\left(\frac{c\rho a R^4}{I_1}\right)$ |
| $\rho$ | air density |

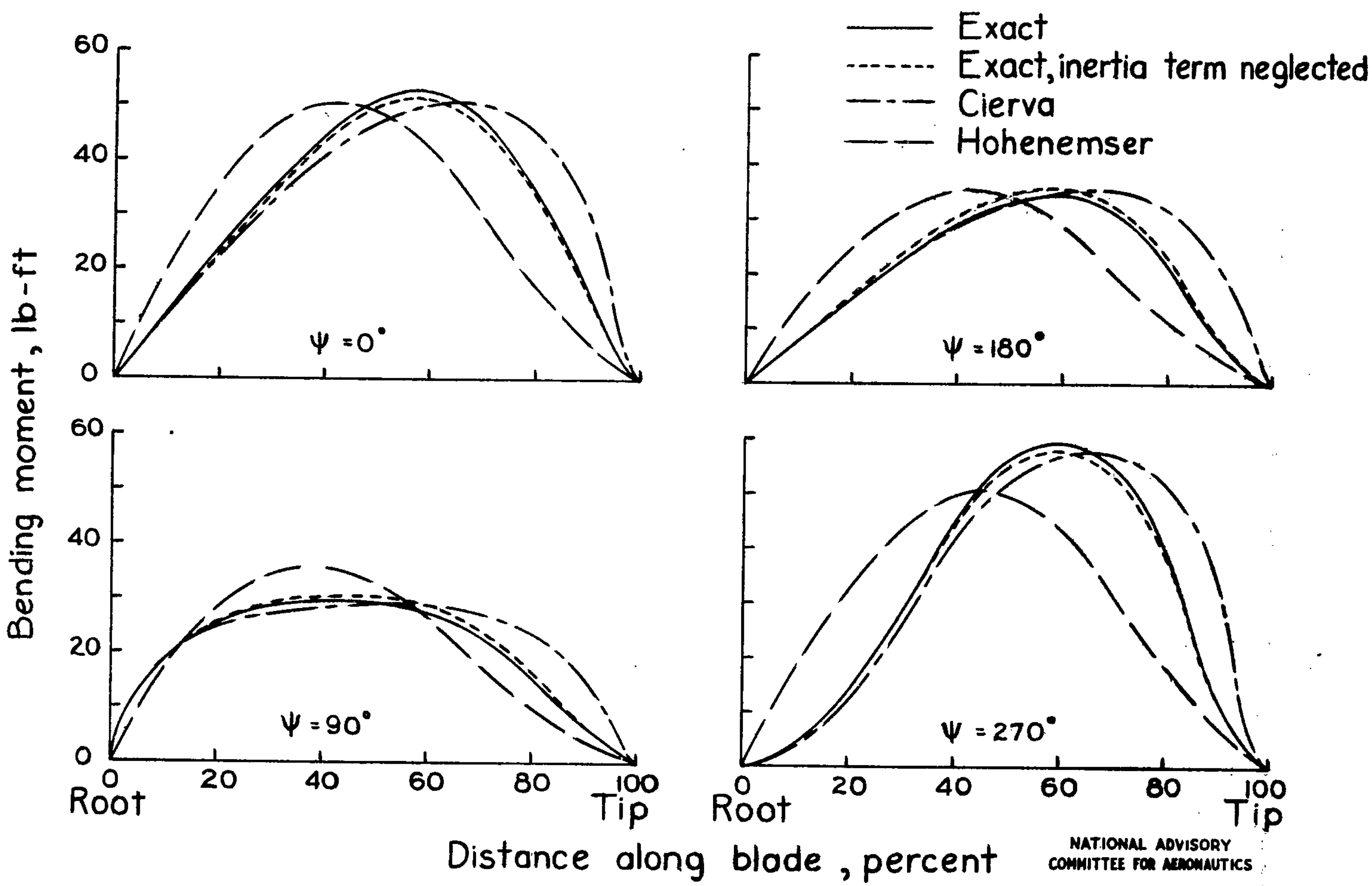


Figure 1.- Comparison of bending-moment analyses.

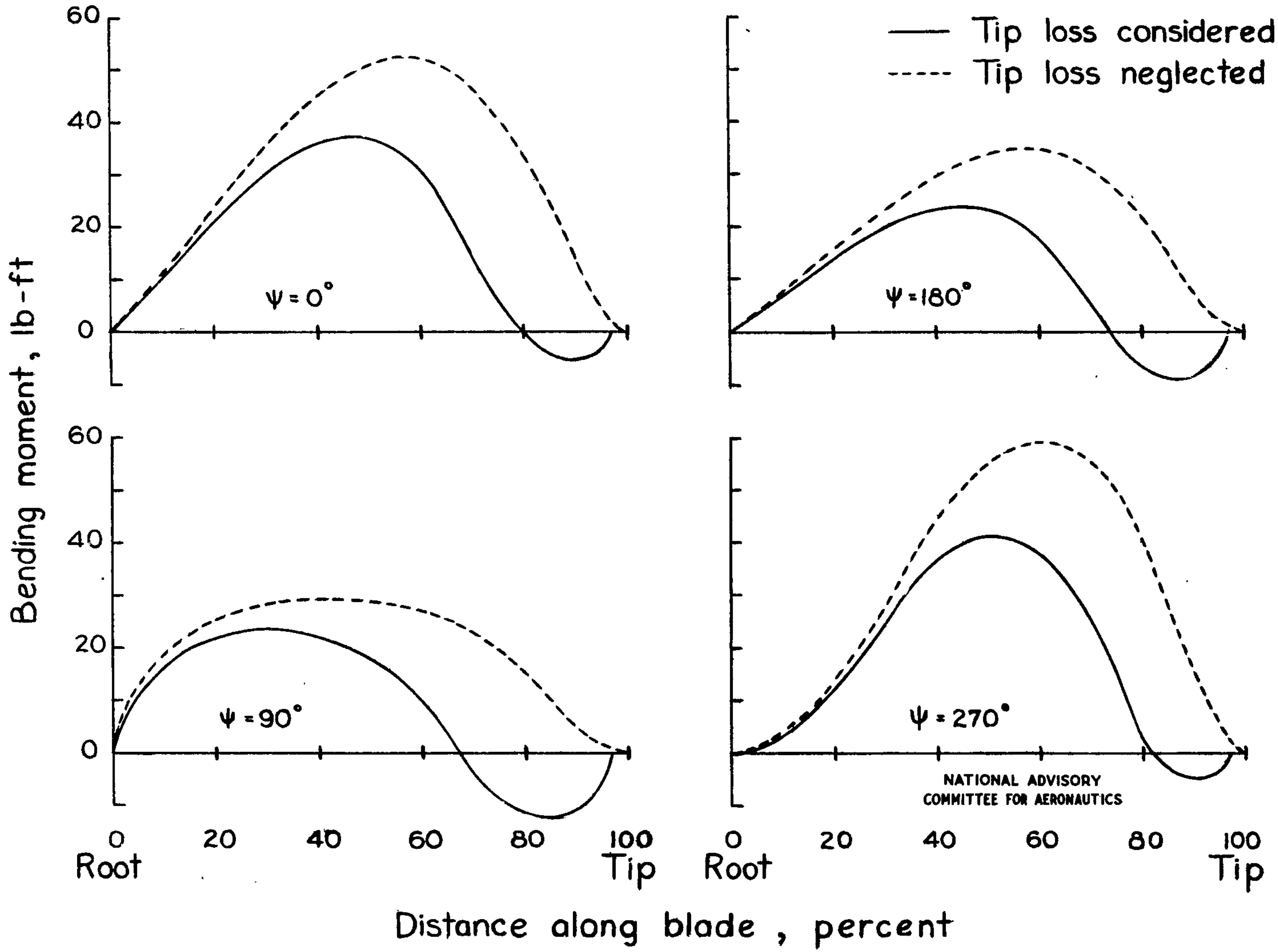


Figure 2.- Effect of tip-loss correction on bending moments computed by exact method.

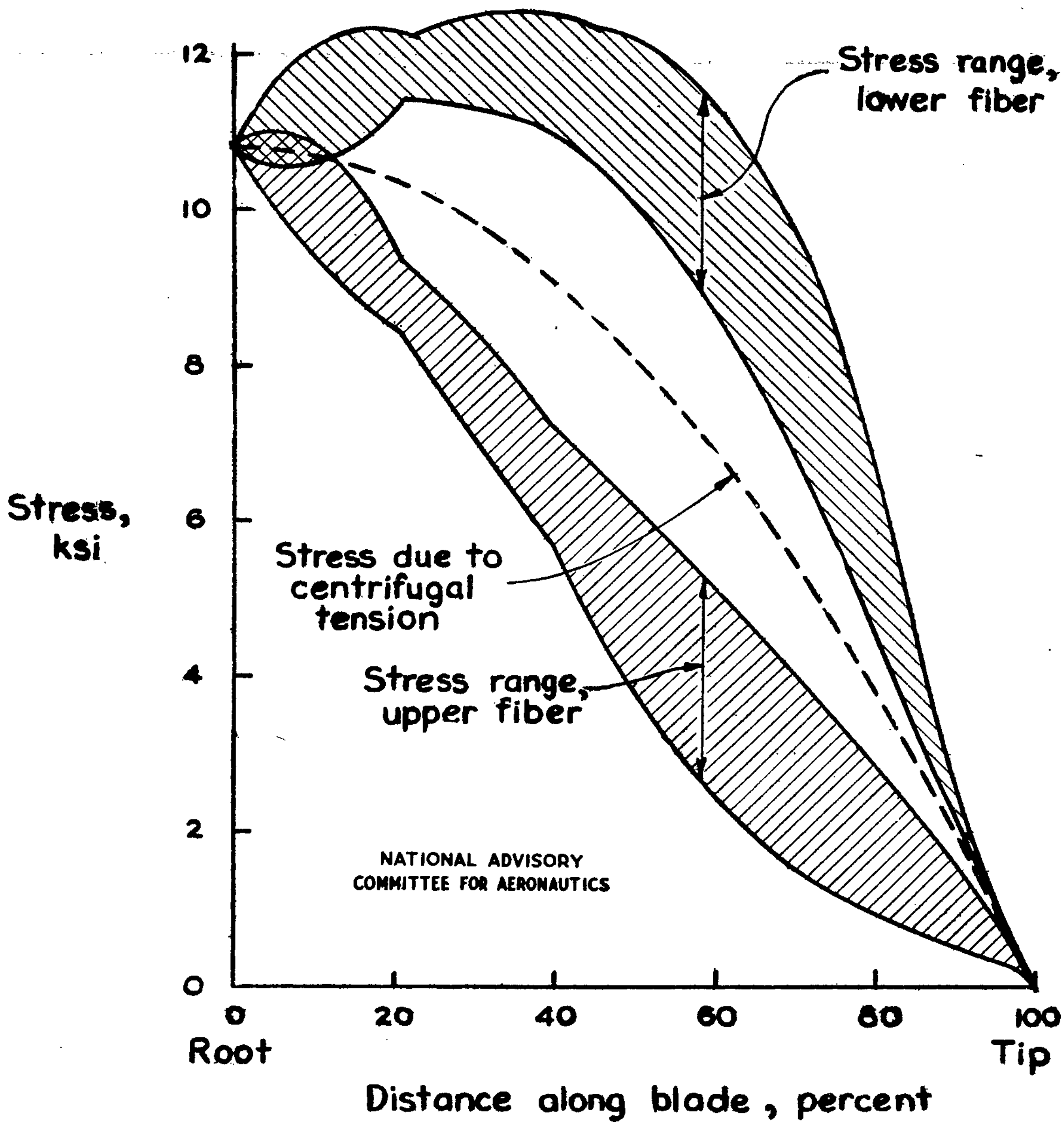


Figure 3.- Stresses due to combined centrifugal tension and bending in plane of flapping.

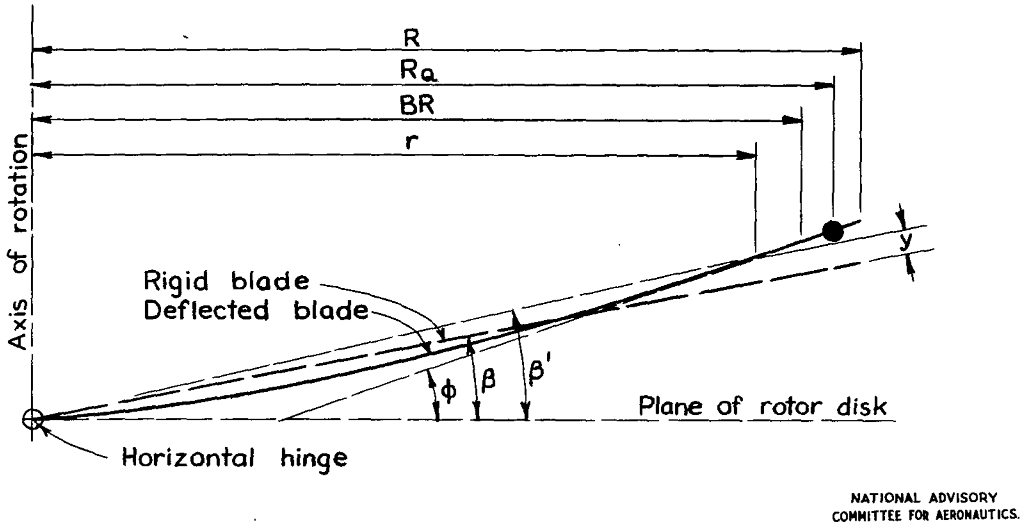


Figure 4.- Geometry of deflected blade.

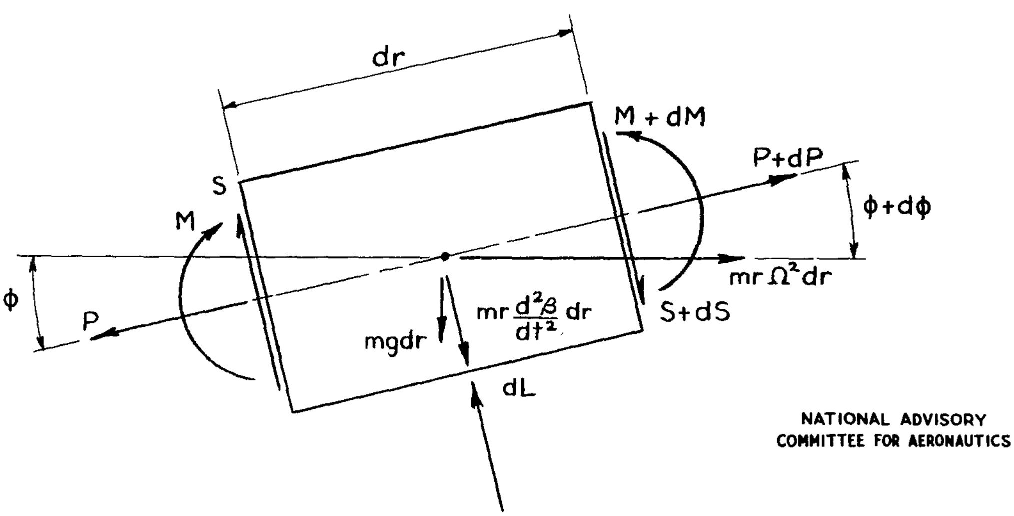


Figure 5.- Forces acting on a blade element in plane of flapping.